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A constraint on the geometry of Yang-Mills theories

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Abstract. *The connection form of a fibre bundle may be identified with the gauge potential of a Yang-Mills theory only if its dependence on the coordinates of the fibres can be eliminated. To maintain this property under a change of bundle coordinates, the standard fibre must admit a globally integrable parallelism. The necessity of principal bundles in the geometrical formulation of Yang-Mills theories can then be deduced.*

1. INTRODUCTION

The classical action for a pure Yang-Mills theory can be regarded as a functional of the curvature form of a principal fibre bundle that is invariant under automorphisms of the bundle which act trivially on the base space [1] [2]. While the set of gauge transformations determine the structure group of the bundle, there seems to be no similar constraint on the standard fibre. The defining property of a principal bundle, that the standard fibre coincide with the structure group [3], is one that could apparently be eliminated. However, attempts to generalize pure Yang-Mills theories by allowing the standard fibre to be a non-group manifold have been unsatisfactory [4] [5]. The aim of this paper is to demonstrate that the geometry is strongly constrained by physical requirements. Of particular importance is the condition that the fields and transformation rules are defined entirely on the base space; otherwise, any gauge invariance of

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the field theory cannot be regarded as an internal symmetry. It will be shown that the independence of the gauge potential with respect to the fibre coordinates can be maintained under gauge transformations only if the standard fibre is a Lie group. One consequence of this result is that the potential must transform under the adjoint representation of the gauge group. A similar conclusion may be reached by considering the consistency of nonlinear self-couplings of massless spin-one fields [6].

2. THE CONNECTION FORM

A connection in a principal bundle (P, M, π, G) (1) provides a decomposition of the tangent space at $p \in P$ into a horizontal subspace H_p and a vertical subspace V_p, tangent to the fibres [7]. The horizontal subspace H_p depends smoothly on p, and it is required to be invariant under right translations $R_{g*}H_p = H_{p\cdot g}$. The diffeomorphism between the fibre passing through p and the group G induces an isomorphism between V_p and the Lie algebra of G. Thus, given a connection, a Lie-algebra-valued connection form can be defined on P by projecting any tangent vector in $T_p(P)$ to V_p and then mapping the vertical component to the Lie algebra. For each local section $\sigma : U \subset M \to \pi^{-1}(U)$, this connection form can be pulled back to the base space by $\Gamma^{(\sigma)} = \sigma^*\Gamma$. In local coordinates for the neighbourhood U, $\Gamma^{(\sigma)} = A_\mu^{(\sigma)}\, dx^\mu$, where $A_\mu^{(\sigma)}$ can be identified as a Lie algebra-valued gauge potential, since, under a change of section, $\sigma(x) \to \sigma'(x) = \sigma(x)g(x)$, it transforms as $A_\mu^{(\sigma')} = Ad(g^{-1})A_\mu^{(\sigma)} + g^{-1}\partial_\mu g$ [1].

Consider now a general fibre bundle (E, M, π). Given a bundle atlas $\{U_\alpha, \psi_\alpha\}$, every point $x \in M$ has a neighbourhood U_α such that ψ_α is a diffeomorphism from $\pi^{-1}(U_\alpha)$ to $U_\alpha \times F$ for some standard fibre F. If $x \in U_\alpha \cap U_\beta$, $y \in F$, then $\psi_\alpha \circ \psi_\beta^{-1}(x, y) = (x, \phi_{\alpha\beta}(x, y))$, where $\phi_{\alpha\beta}(x, \cdot) \in Diff(F)$ are transition functions.

The trivializations ψ_α, ψ_β determine two local sections $\sigma_\alpha(x) = \psi_\alpha^{-1}(x, y_0)$, $\sigma_\beta(x) = \psi_\beta^{-1}(x, y_0)$. If $U \subset U_\alpha \cap U_\beta$, an arbitrary trivialization $\psi : \pi^{-1}(U) \to U \times F$ will map $\sigma_\alpha(x)$, $\sigma_\beta(x)$ to $\psi\sigma_\alpha(x) = (x, y(x))$, $\psi\sigma_\beta(x) = (x, y'(x))$. Let us define $\psi_{\beta\alpha} : U \times F \to F$ by $\psi_{\beta\alpha}(x, y) \equiv \psi_{\beta\alpha}^y(x) \equiv \psi_{\beta\alpha}^x(y) = y'$, where $\psi_{\beta\alpha}^x \in Diff(F)$. Then the tangent spaces to the two sections are related by

$$(1) \qquad (\xi_x, V^\beta) \equiv (\psi\sigma_\beta)_*(\xi_x) = (\xi_x, T\psi_{\beta\alpha} \cdot (\psi\sigma_\alpha)_*(\xi_x)) = (\xi_x, T\psi_{\beta\alpha}(\xi_x, V^\alpha))$$

Any bundle (E, M, π) for which the total space E is a paracompact manifold admits a C^r connection, $0 \leqslant r \leqslant \infty$ [8]. The tangent bundle is then the direct sum

(1) P is the total space, M is the base space, π is a smooth projection of P onto M, and G is the structure group.

of the vertical and horizontal subbundles, $TE = VE \oplus HE$. The horizontal projection of the tangent vectors $\sigma_{\alpha *} \cdot \xi_x$, $\sigma_{\beta *} \cdot \xi_x$ may be mapped onto $TU \times TF$, with the components along TF denoted by C^α, C^β respectively. The horizontal subspaces $\psi_* H_{\sigma_\alpha(x)}$, $\psi_* H_{\sigma_\beta(x)}$ are spanned by vectors of the form $(\xi_x, C^\alpha(y))$, $(\xi_x, C^\beta(y'))$, as the components along TF must independent of ξ_x. Thus, the diffeomorphism $\psi^x_{\beta\alpha}$ induces a mapping from $\psi_* H_{\sigma_\alpha(x)}$ to $\psi_* H_{\sigma_\beta(x)}$, so that

$$C^\beta(y') = T\psi^x_{\beta\alpha} \cdot C^\alpha(y) \tag{2}$$

The connection forms Γ^α, Γ^β are obtained by subtracting C^α, C^β from V^α, V^β. The transformation from Γ^α to Γ^β is given by

$$\begin{aligned} \Gamma^\beta(\xi_x, y') &= V^\beta(\xi_x, y') - C^\beta(\xi_x, y') \\ &= T\psi^y_{\beta\alpha}(\xi_x) + T\psi^x_{\beta\alpha} \cdot V^\alpha(\xi_x, y) - T\psi^x_{\beta\alpha} \cdot C^\alpha(\xi_x, y) \\ &= T\psi^y_{\beta\alpha}(\xi_x) + T\psi^x_{\beta\alpha} \cdot \Gamma(\xi_x, y) \end{aligned} \tag{3}$$

The transformation rule of the connection form for the bundle (E, M, π) generally involves the fibre coordinate.

3. INDEPENDENCE OF GAUGE TRANSFORMATION WITH RESPECT TO THE FIBRE COORDINATE

To identify the connection forms with gauge potentials depending only on the coordinates of the base space M, one must extend $\Gamma^\alpha(\xi_x, y) \in T_y(F)$ and $\Gamma^\beta(\xi_x, y') \in T_{y'}(F)$ to global vector fields on F. For any $y \in F$, define a mapping ϕ_y from a vector space V to the tangent space $T_y(F)$, such that $\Gamma^\alpha(\xi_x, y) = \phi_y \Gamma^\alpha(\xi_x)$ where $\Gamma^\alpha(\xi_x) \in V$.

PROPOSITION 1. *The dependence on the fibre coordinate in the transformation rule for the connection form can be eliminated only if the fibre F admits a parallelism.*

Proof. It follows from (3) that

$$\phi_{y'} \Gamma^\beta(\xi_x) = T\psi^y_{\beta\alpha}(\xi_x) + T\psi^x_{\beta\alpha} \cdot \phi_y \Gamma^\alpha(\xi_x) \tag{4}$$

The relation between $\Gamma^\alpha(\xi_x)$ and $\Gamma^\beta(\xi_x)$ can be interpreted as a gauge transformation if the fibre coordinate can be eliminated from (4). As both terms on the right-hand side of (4) can be expressed as $\phi_{y'} X$, $X \in V$, ϕ_y should be a vector space isomorphism which depends smoothly on y. Thus, $\phi : V \times F \to TF$, $\phi(\cdot, y) = \phi_y$, represents a parallelism on F, with V being the reduced tangent space [9]. ∎

Now consider the second term on the right-hand side of (4). To eliminate the fibre coordinate, we require

$$T\psi^x_{\beta\alpha} \cdot \phi_y = \phi_{y'} \cdot A(x) \tag{5}$$

where $A(x) : V \to V$ is a linear isomorphism depending only on the $\psi^x_{\beta\alpha}$ and not the fibre coordinate y. For a Yang-Mills theory associated with a principal bundle (P, M, π, G) V is the Lie algebra of G and $A(x) = Ad(g^{-1}(x))$, $g(x) \in G$. When $A(x) = Id_V$, equation (5) becomes $T\psi^x_{\beta\alpha} = \phi'_y \cdot \phi_y^{-1}$. The parallelism ϕ defines a differential system, whose solution is a local diffeomorphism mapping y to y'. The parallelism is defined to be integrable if a solution exists for any pair of points $y, y' \in F$. When every solution can be extended to a global diffeomorphism on F, the parallelism is globally integrable, and F is a Lie group [9].

Even if $A(x) \neq Id_V$, it can nevertheless be demonstrated that the parallelism must be globally integrable.

PROPOSITION 2. *For any pair of points $y_0, y'_0 \in F$, there exists a global diffeomorphism $\psi^x_{\beta\alpha} : F \to F$, $\psi^x_{\beta\alpha}(y_0) = y'_0$, satisfying $T\psi^x_{\beta\alpha} \cdot \phi_y = \phi_{y'} . A$ for some constant matrix A if and only if the parallelism on F is globally integrable.*

Proof. The global diffeomorphism $\psi^x_{\beta\alpha}$ is strongly constrained by (5). Indeed, by Frobenius' theorem, a local diffeomorphism obeying (5) will exist if and only if

$$[\phi_{y'} A\phi_y^{-1} \cdot X, \phi_{y'} A\phi_y^{-1} \cdot Y]_{y'_0} = \phi_{y'_0} A\phi_{y_0}^{-1} \cdot [X, Y]_{y_0} \tag{6}$$

for any two vector fields X, Y defined in a neighbourhood of y_0. The parallelism induces a set of smooth, linearly independent vector fields $\{\xi_i(y) = \phi_y e_i\}$ where the $\{e_i\}$ form a basis for the vector space V. Expanding the commutator of two vectors ξ_i, ξ_j at y in the basis $\{\xi_i(y)\}$, $[\xi_i, \xi_j]_y = c_{ijk}(y)\xi_k(y)$, and noting that

$$\phi_{y'} A\phi_y^{-1} \xi_i(y) = \phi_{y'} A e_i = A_{ki}\xi_k(y') \tag{7}$$

the integrability condition (6) becomes

$$c_{ijk}(y_0)A_{mk} = c_{klm}(y'_0)A_{ki}A_{lj} \tag{8}$$

This identity holds for all points in a neighbourhood of y_0 when $\psi^x_{\beta\alpha}$ is a local diffeomorphism. If we require $\psi^x_{\beta\alpha}$ to be a global diffeomorphism obeying (5), then (8) must be satisfied at every point on the fibre. It will now be shown that no constant matrix A can satisfy (8) when the coefficients c_{ijk} depend on the fibre coordinate y. This is a consequence of (8) leading to an infinite number of independent constraints on the constant matrix.

Suppose the contrary. Then for any pair of points $(y, \tilde{y})$, there is a global diffeomorphism β such that $T\beta \cdot \phi_y = \phi_{\tilde{y}} \cdot B$

$$c_{ijk}(y)B_{mk} = c_{klm}(\tilde{y})B_{ki}B_{lj} \tag{9}$$

If y is chosen to be different from y', $c_{ijk}(\tilde{y}) \neq c_{ijk}(y')$, it may be assumed that B does not obey (8). Let us define the map γ by $\psi^x_{\beta\alpha}(\tilde{y}) = \psi^x_{\beta\alpha} \cdot \beta(y) \equiv \gamma \cdot \cdot \psi_{\beta\alpha}(y) = \gamma(y')$. Then $T\gamma \cdot \phi_{y'} = \phi_{\psi^x_{\beta\alpha}(y)} \cdot C$, $C = ABA^{-1}$. From (8) and (9),

$$\begin{aligned} c_{ijk}(\tilde{y}_0)A_{mk} &= c_{klm}(\psi^x_{\beta\alpha}(\tilde{y}_0))A_{ki}A_{lj} \\ &= [c_{lno}(y_0)(B^{-1})_{li}(B^{-1})_{nj}B_{ko}]A_{mk} \\ &= [c_{pqr}(y'_0)(C^{-1})_{pk}(C^{-1})_{ql}C_{mr}]A_{ki}A_{lj} \end{aligned} \tag{10}$$

If $[A, B] = 0$, then $B = C$ and (10) becomes

$$[c_{lno}(y_0)A_{ko}](B^{-1})_{li}(B^{-1})_{nj}B_{mk} = [c_{pqr}(y'_0)A_{pk}A_{ql}](C^{-1})_{ki}(C^{-1})_{lj}C_{mr} \tag{11}$$

which is equivalent to (8). Otherwise (10) represents an independent condition on A.

Let F be a smooth n-dimensional manifold, so that it may be covered by open patches which are homeomorphic to open sets in R^n. Since the tangent space at any point on F is isomorphic to R^n, the reduced tangent space V can be identified with R^n. If there is a basis of V, $(e_1, \ldots e_n)$, in which A is a diagonal matrix with n distinct eigenvalues, the matrices commuting with A are also diagonal. If the eigenvalues of A are not all different, then there will be matrices with off-diagonal elements commuting with A. However, these matrices will not necessarily commute with each other, and, in fact, the maximal commuting subgroup in this set is again the group of diagonal matrices. When A has complex eigenvalues and cannot be diagonalized over R^n, the argument above can be adapted to show that the maximal abelian group of linear transformations of V commuting with A is still n-dimensional.

However, not all matrices in this group can satisfy an identity such as (9). This is demonstrated in the following lemma.

LEMMA 1. *Suppose there is a diffeomorphism* $\beta(\lambda)$ *such that* $T\beta(\lambda) \cdot \phi_y = \phi_{\tilde{y}(\lambda)} \cdot \lambda B$, and

$$c_{ijk}(y)\lambda B_{mk} = c_{klm}(y(\lambda))\lambda B_{ki}\lambda B_{lj} \tag{12}$$

Then λ *must equal 1.*

Proof. It follows from (9) and (12) that $c_{klm}(y) = \lambda c_{klm}(\tilde{y}(\lambda))$. The coefficients c_{klm} are also constrained by the following property. If X, Y, Z are three parallel vector fields on F, so that their components with respect to the basis $\{\xi_i(y)\}$ are constant, then $[[X, Y], Z]$ is also a parallel vector field [10]. Thus for any $y, y' \in F$

(13) $$[[\xi_i, \xi_j], \xi_k]_{y'} = \phi_{y'} \cdot \phi_y^{-1}[[\xi_i, \xi_j], \xi_k]_y$$

or equivalently,

(14) $$c_{ijl}(y')c_{klm}(y') - \xi_k(c_{ijm})(y') = c_{ijl}(y)c_{klm}(y) - \xi_k(c_{ijm})(y)$$

In particular, for $\tilde{y}$, $\tilde{y}(\lambda) = \beta(\lambda) \cdot \beta^{-1}(\tilde{y})$, (14) gives

(15) $$\lambda^{-2}c_{ijl}(\tilde{y})c_{klm}(\tilde{y}) - \lambda^{-1}\xi_k(c_{ijm})(\tilde{y}) = c_{ijl}(\tilde{y})c_{klm}(\tilde{y}) - \xi_k(c_{ijm})(y)$$

There are two solutions to this equation, but by again considering (14) for the pair y, $(\beta(\lambda) \cdot \beta^{-1})^{-2}(y)$, we see that $\lambda = 1$ is the only acceptable one.

From this lemma, it follows that the set of matrices commuting with A and satisfying (9), to be denoted as M_A, has maximum dimension $n - 1$. Now fix a point $y_0 \in F$. If $T\beta(y_0, y) \cdot \phi_{y_0} = \phi_y \cdot B_y$, then $y \to B_y$ is a mapping of F onto M_A. This mapping is surjective, but we will now show that its domain cannot include all of F. Consider two points $y_i = \beta_i(y)$, $i = 1, 2$, such that $T\beta_i \cdot \phi_{y_0} = \phi_{y_i} \cdot B_i$. When $B_1 = B_2$, $T(\beta_2 \cdot \beta^{-1}) \cdot \phi_{y_1} = \phi_{y_2}$, which implies $c_{ijk}(y_1) = c_{ijk}(y_2)$. The diffeomorphism $\beta_2 \cdot \beta_1^{-1}$ is known as a translation, and the set of translations forms a pseudogroup [9]. If the pseudogroup is transitive on F, then c_{ijk} is constant and the parallelism is integrable. By our original assumption, however, the coefficients c_{ijk} are not constant and the set $C_{y_1} = \{y \in F \mid c_{ijk}(y) = c_{ijk}(y_1)\}$ is strictly contained in F. It will now be shown that this assumption cannot hold. ∎

LEMMA 2. *The set C_{y_1} contains all of F.*

Proof. Suppose initially that C_{y_1} is a continuous curve passing through y_1. Let X_{y_1} be the tangent vector at y_1 so that $X(c_{ijm})(y_1) = X_k \xi_k(c_{ijm})(y_1) = 0$. By the identity (14), $X(c_{ijm})(y) = 0$ for all y on the curve C_{y_1}. Thus C_{y_1} is an integral curve of the parallel vector field X.

The derivative of c_{ijk} in the direction of an arbitrary tangent vector Y also takes the same value at every point on C_{y_1}. Thus, if $\{\phi_t\}$ is the one-parameter family of diffeomorphisms generated by the vector field Y, the translation of C_{y_1} by a distance t along the integral curves of Y gives a curve of constant c_{ijk}, $C_{\phi_t(y_1)}$. The tangent vector to $C_{\phi_t(y_1)}$ is

(16) $$\phi_{t*}X = X - t[X, Y] + \frac{t^2}{2}[[X, Y], Y] - \ldots$$

Associated to each vector field Y is a two-dimensional surface S_Y spanned by the integral curves of Y passing through C_{y_1}. It can be shown that there will exist different surfaces $S_{Y'}$, $S_{Y''}$ that intersect at a non-zero distance from C_{y_1}.

Suppose $Y_1, \ldots, Y_{n-1}$ are independent vector fields such that the integral curves of linear combinations of the Y_i do not intersect and sweep out an n-dimensional neighbourhood of y_1. If $Y = \sum_i a_i Y_i$ and $Y' = aX + bY$, the surfaces $S_Y, S_{Y'}$ initially coincide at C_{y_1}. However, upon substituting Y' in (16), one sees that the tangent spaces to $S_Y, S_{Y'}$ will differ elsewhere as they are spanned by linearly independent vectors. Since $S_{Y'}$ does not coincide with S_Y everywhere, it must intersect another surface $S_{Y''}$. Let y_3 be a point in this intersection. The set C_{y_3} consists of a curve in $S_{Y'}$ with tangent vector $X - t'[X, Y'] + \frac{t^2}{2}[[X, Y'].\ Y'] - \ldots$ and curve in $S_{Y''}$ with tangent vector $X - t''[X, Y''] + \frac{t''^2}{2}[[X, Y''], Y''] - \ldots$ where t' t'' are parameters for the integral curves of Y', Y'' respectively. These vectors are equal if they agree to each order in the infinitesimally small parameters t', t'', which is only possible if Y'' is a multiple of Y'. Since $Y'' \neq cY'$, the tangent vectors do not coincide, and by linearity, they generate a two-dimensional surface of constant c_{ijk}. By translating this surface back to y_1, one now finds that C_{y_1} is two-dimensional. Upon repetition of the above argument sufficiently many times, it can be concluded that c_{ijk} is constant on an n-dimensional neighbourhood of y_1, U_{y_1}. Since F can be covered by overlapping neighbourhoods U_y, c_{ijk} must be constant on all of F.

Since $\{\xi_i(y)\}$ are smooth vector fields, the coefficients c_{ijk} are smooth functions and the only remaining possibility is that C_y is a set of isolated points. However, if C_y is zero-dimensional and M_A has maximum dimension $n-1$, there are infinitely many points in F which cannot be mapped onto M_A. Corresponding to each of these points is an independent constraint on A. As A is a constant matrix, not all of these constraints can be satisfied. Therefore, C_{y_1} is n-dimensional and contains all of F. ■

As a consequence of Lemma 2, it follows that the coefficients c_{ijk} are constant and the parallelism on F must be globally integrable. The fibre F is then of the form G/D, where G is a Lie group and D is a discrete subgroup [10]. Assuming F is simply connected, it will be a Lie group.

Conversely, when the parallelism is globally integrable and F is a Lie group, the coefficients c_{ijk} are constant, and equation (8) will have solutions. The simplest solution is $A_{mk} = \delta_{mk}$. More generally, one may note that the reduced tangent space V can be identified with the Lie algebra, and given an arbitrary group element $g = \exp(t_j\xi_j)$, $A = Ad(g)$ is an automorphism of V

$$(17) \qquad Ad(g)\xi_i = \exp(t_j\xi_j)\xi_i \exp(-t_j\xi_j) = \xi_m A_{mi}$$

where

$$A_{mi} = \delta_{mi} + t_j c_{ijm} + \frac{1}{2} t_j t_k c_{ikl} c_{ljm} + \ldots \tag{18}$$

Using the Jacobi identity for the structure constants, it can be verified to each order in t that this expression for A_{mi} represents the most general solution to (8). This completes the proof of Proposition 2. ∎

On Lie groups, there are two types of parallelism, induced by left and right multiplication. We will set $\phi_y = L_{y*}$. Since

$$\begin{aligned} L_{g*} L_{y*} &= L_{g \cdot y*} \cdot Id \\ R_{g*} L_{y*} &= L_{y \cdot g*} \cdot Ad(g^{-1}) \end{aligned} \tag{19}$$

both left and right translations are diffeomorphisms of the fibre satisfying (5). In fact, these are the only such diffeomorphisms, because any mapp [Aing ψ, whose tangent is $T\psi = \phi_{\psi(y)} \cdot Ad(g^{-1}) \cdot \phi_y^{-1}$, leaves the metric on the group manifold G invariant. Thus ψ must be an isometry, and the isometry group of G is $G \times G$.

Let us now consider the y-dependence of the first term on the right-hand side of (4).

PROPOSITION 3. *Let ϕ_y be the parallelism on G induced by left multiplication. Then $T\psi^y_{\beta\alpha}(\xi_x) = \phi_{y'}(x)$, $X(x) \in V$, only when $\psi^x_{\beta\alpha} = R_{g(x)}$.*

Proof. For left translations,

$$\psi^x_{\beta\alpha}(y) = L_g(y) = g(x) \cdot y = R_y \cdot g(x) = \psi^y_{\beta\alpha}(x) \tag{20}$$

defines $\psi^y_{\beta\alpha}$. The tangent mapping is

$$\begin{aligned} T\psi^y_{\beta\alpha}(\xi_x) &= R_{y*} \cdot (g^* \cdot \xi_x) = R_{y*} R_{g*} [R^{-1}_{g*}(g^* \cdot \xi_x)] \\ &= \phi_{y'} Ad(y'^{-1}) [R^{-1}_{g*}(g^* \cdot \xi_x)] \end{aligned} \tag{21}$$

The dependence of this term on the fibre coordinate is not eliminated after multiplication by $\phi^{-1}_{y'}$. For right translations, $\psi^y_{\beta\alpha}(x) = L_y \cdot g(x)$, so that

$$T\psi^y_{\beta\alpha}(x) = L_{y*}(g^* \cdot \xi_x) = \phi_{y'}[L^{-1}_{g*}(g^* \cdot \xi_x)] \tag{22}$$

Equation (4) now becomes

$$\Gamma^\beta(\xi_x) = Ad(g^{-1}) \Gamma^\alpha(\xi_x) + L^{-1}_{g*}(g^* \cdot \xi_x) \tag{23}$$

which takes the standard form for a gauge transformation upon setting ξ_x equal to ∂_μ. If the parallelism $\phi_y = R_{y*}$ is chosen, a similar result is obtained when $\psi^x_{\beta\alpha} = L_{g(x)}$. ∎

If G is abelian, the structure constants vanish and the matrix A is unconstrained. Consequently, the y-dependence of $T\psi^x_{\beta\alpha} \cdot \phi_y \Gamma^\alpha(\xi_x)$ does not directly lead to a restriction on the diffeomorphism $\psi^x_{\beta\alpha}$. The precise form of $T\psi^y_{\beta\alpha}(\xi_x)$ is therefore undetermined. However, one has the following result.

PROPOSITION 4. *If G is abelian, and* $\phi_y = L_{y*} = R_{y*}$, $\psi^x_{\beta\alpha} = L_{g(x)} = R_{g(x)}$.

Proof. Define the following maps $M \to Diff(G) \to G$ by

$$\begin{aligned} &\psi_{\beta\alpha}(x) \equiv \psi^x_{\beta\alpha} \\ &O_y \cdot \psi_{\beta\alpha}(x) = \psi^y_{\beta\alpha}(x) \end{aligned} \tag{24}$$

To obtain a y-independent transformation rule, we require the tangent mappings to obey

$$T\psi^y_{\beta\alpha}(\xi_x) = TO_y \cdot T\psi_{\beta\alpha}(\xi_x) = \phi_{y'}X(x) \tag{25}$$

where $y' = \psi^x_{\beta\alpha}(y)$ and $X(x)$ is an element of the Lie algebra of G. The projection operator O_y can always be expressed as $L_y\pi(y)$ where $\pi(y) : Diff(G) \to G$ is defined by

$$\pi(y) \cdot \psi_{\beta\alpha}(x) = L_y^{-1}\psi^x_{\beta\alpha}(y) \tag{26}$$

Equation (25) is then equivalent to

$$\phi_{y'}^{-1}\phi_y\pi(y)_*T\psi_{\beta\alpha}(\xi_x) = X(x) \tag{27}$$

From (7), we recall that

$$T\psi^x_{\beta\alpha} \cdot \xi_i(y) = \phi_{y'}A\phi_y^{-1}\xi_i(y) = A_{ki}\xi_k(y') \equiv A\phi_{y'}\phi_y^{-1}\xi_i(y) \tag{28}$$

which implies $(T\psi^x_{\beta\alpha})^{-1}A = \phi_y\phi_{y'}^{-1}$, where the matrix A is independent of y by definition. Since the group is abelian, $\phi_y\phi_{y'}^{-1} = \phi_{y'}^{-1}\phi_y$ and

$$(T\psi^x_{\beta\alpha})^{-1}A\pi(y)_*T\psi_{\beta\alpha}(\xi_x) = X(x) \tag{29}$$

This condition will hold only if π is independent of y, which implies that $\psi^x_{\beta\alpha}(y) = y \cdot g(x) = R_g(y)$. ∎

4. CONCLUSION

For any Lie group G, if the parallelism ϕ_y is induced by left translations, the only allowed coordinate transformations are the right translations $R_{g(x)}$. Consequently, the structure group of the bundle (E, M, π) is reduced to G. One may also note from equation (2) that the distribution of horizontal subspaces in TE is invariant under right translations, $H_{p \cdot g} = R_{g*}H_p$, $p \in E$. This property is required for a connection in a principal bundle, and the existence of such a connection can be proven [3].

To summarize, the geometry of a pure Yang-Mills theory is described by the principal fibre bundle, with the connection form being identified as the gauge potential. A general bundle (E, M, π), where E is paracompact, also admits a connection and the transformation rule for the connection form under a change of bundle coordinates is given by equation (3). This connection form can only be regarded as a gauge potential if the dependence of the transformation on the fibre coordinates can be eliminated. By analyzing the y-dependence of both terms on the right-hand side of equation (3), one obtains strong restrictions on the standard fibre F, which is initially assumed to be an arbitrary smooth, simply-connected, finite-dimensional manifold. In particular, the fibre F must admit a globally integrable parallelism, which implies that it is a Lie group. The structure group is similarly constrained. Even when F is a Lie group, the allowed bundle transformations consist of either right or left multiplication but not both. Thus, the structure group and the standard fibre coincide, and (E, M, π) must be a principal bundle.

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